2ND "MATH-CHALLENGE" PRIZE PROBLEMS: PURE MATHEMATICS

ABSTRACT. This is the first problem set of a series of mathematical challenges with prizes ranging from GHC 300.00 to GHC 700.00 to be awarded per **rigorous** solution to each of the problems. Detailed rules governing the problems and the prizes can be found at the announcement page at **https://math.knust.edu.gh**.

1. Intermediate Level Problems

These are worth GHC 300.00[1] per rigorous solution. The deadline for submitting a solution is 23:59 GMT Thursday 31st October, 2019.

MaTH-Challenge Problem 1A (Linear Algebra). *Consider the system of 4 linear equations written in matrix form below:*

$$\begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Demonstrate the conditions on the numbers a, b, c, d such that the system of equations cannot be solved for a unique value of w, x, y, z and determine the values of the special cases for which $a = 2$, $b = 0$, $0 \leq c \leq 9$ and $1 \leq d \leq 9$.

MaTH-Challenge Problem 1B (Geometry). *Let $\mathcal{C}$ be any simple closed curve in the Cartesian plane (i.e., $\mathcal{C}$ is a closed curve that does not intersect itself). Show that given three vertices of any triangle T in the Cartesian plane, there exists another triangle T^* that is similar to T but whose three vertices lie exactly on $\mathcal{C}$.*

MaTH-Challenge Problem 1C (Number Theory). *Let a, b, c be some given integers and x, y, z be variables. Suppose the equation*

$$ax^2 + by^2 = cz^2$$

has a solution $x = x_0, y = y_0, z = z_0$ that are all integers. Show that there are infinitely many other integer solutions that solve the equation above, and give the formula for these solutions in terms of x_0, y_0, z_0.

[1]Subject to funding availability, this prize money may be increased to GHC 500.00

2. Advanced Level Problems

These are worth GHC 700.00[2] per rigorous solution. The deadline for submitting a solution is 23:59 GMT Thursday 14th November, 2019.

MaTH-Challenge Problem 1D (Analysis)**.** *Let $\mathbb{R}^n$ be the Euclidean space of dimension n and let $\mathcal{S} \subset \mathbb{R}^n$ be any closed and bounded star-domain (i.e., there is at least one point $\boldsymbol{u} \in \mathcal{S}$ such that $t\boldsymbol{u}+(1-t)\boldsymbol{x} \in \mathcal{S}$ for all $t \in [0,1]$ and $\boldsymbol{x} \in \mathcal{S}$). Suppose $f\colon \mathcal{S} \to \mathcal{S}$ is a metric map (i.e. the distance between $f(\boldsymbol{x})$ and $f(\boldsymbol{y})$ is never greater than the distance between $\boldsymbol{x}$ and $\boldsymbol{y}$ for all $\boldsymbol{x},\boldsymbol{y} \in \mathcal{S}$). Show that there exists at least one vector in $\mathcal{S}$ that is invariant (in other words, fixed) by f.*

MaTH-Challenge Problem 1E (Topology)**.** *Let $\mathbb{C}$ be the topological space of the complex numbers with the topology generated by the metric induced by the usual absolute value on $\mathbb{C}$, and let $\mathbb{S} \subset \mathbb{C}$ be the unit circle (i.e. set of complex numbers of absolute value 1). Suppose f is an embedding of $\mathbb{S}$ into $\mathbb{C}$ (i.e., $f\colon \mathbb{S} \to \mathbb{C}$ is a homeomormorphism). Show that there exists $w, z \in \mathbb{S}$ and $a, b \in \mathbb{C}$, with $b \neq 0$, such that the numbers $a + bz, a + b\bar{z}, a - bz, a - b\bar{z}$ all belong to the image of f.*

MaTH-Challenge Problem 1F (Number Theory)**.** *Let $\{s_n\}_{n\geq 1}$ be the sequence given by the recursion*

$$n^3 s_{n+2} - (34n^3 - 51n^2 + 27n - 5)s_{n+1} + (n-1)^3 s_n = 0\,,$$

with the first two values of the sequence stipulated as $s_1 = 10^{10^{10}}$ and $s_2 = 1$. Show that the values of the sequence are all integers (i.e., for every natural number n, the value of s_n is an integer).

3. Rules for the MaTH-Challenge Prize

Solutions to the MaTh-Challenge Problems should be sent to the email address below (depending on the problem number). However, detailed rules governing the problems and the prizes can be found at the announcement page at **https://math.knust.edu.gh**.

E-mail address: `jezearn@knust.edu.gh, (Problems 1A,1B,1C,1D,1E,1F)`
E-mail address: `rkboadi.cos@knust.edu.gh, (Problems 1B,1C,1D,1F)`
E-mail address: `b.bainson@aims.edu.gh, (Problems 1A,1E)`

[2]Subject to funding availability, this prize money may be increased to GHC 1000.00